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On Piterbarg theorem for maxima of stationary Gaussian sequences

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Abstract. Limit distributions of maxima of dependent Gaussian sequence are different according to the convergence rate of their correlations. For three different conditions on convergence rate of the correlations, in this paper, we establish the Piterbarg theorem for maxima of stationary Gaussian sequences.

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1 Introduction

For a strictly stationary sequence $\{X_n,\, n \geqslant 1\}$, the seminal paper [10] derived joint limiting distributions of maxima of complete and incomplete samples. The sample is often incomplete since observations are missing, which is formalized by introducing a sequence of indicator random variables $\{\varepsilon_n,\, n \geqslant 1\}$, where $\{\varepsilon_n = 1\}$ means that X_n is observed, whereas $\{\varepsilon_n = 0\}$ corresponds to the case X_n is missing. Throughout this paper, $\{\varepsilon_n,\, n \geqslant 1\}$ are independent of the stationary process $\{X_n,\, n \geqslant 1\}$. If F denotes the common distribution function of all X_ns, then the maxima of incomplete sample $M_n(\varepsilon)$, $n \geqslant 1$, is defined by

$$M_n(\varepsilon) = \begin{cases} \max\{X_j,\ \varepsilon_j = 1,\ j \leqslant n\} & \text{if } \sum_{j=1}^n \varepsilon_j \geqslant 1, \\ \inf\{t\colon\ F(t) > 0\} & \text{otherwise.} \end{cases}$$

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It is well known (see, e.g., [3]) that if, for some norming constants $a_n > 0$ and $b_n \in \mathbb{R}$,

$$\lim_{n\to\infty} F^n(a_n x + b_n) = G(x) \quad \forall x \in \mathbb{R} \tag{1.1}$$

with G being an extreme value distribution of some random variable $\mathcal{M}$, then

$$\lim_{n\to\infty} \mathbf{P}\bigl\{a_n^{-1}(M_n - b_n) \leqslant x\bigr\} = G(x) \quad \forall x \in \mathbb{R} \tag{1.2}$$

with $M_n = \max_{1\leqslant i\leqslant n} X_i$ under some additional weak dependence conditions. Assume that, for some constant $\mathcal{P} \in [0,1]$, the indicator random sequence $\{\varepsilon_n,\ n \geqslant 1\}$ satisfies

$$\frac{S_n}{n} := \frac{\sum_{i=1}^n \varepsilon_i}{n} \to \mathcal{P} \quad \text{in probability} \tag{1.3}$$

as $n \to \infty$. The contribution [10] proved that under the conditions $D(u_n, v_n)$ and $D'(u_n)$,

$$\lim_{n\to\infty} \mathbf{P}\bigl\{M_n(\varepsilon) \leqslant u_n,\ M_n \leqslant v_n\bigr\} = H(\mathcal{P}, x, y) =: G^{\mathcal{P}}(x) G^{1-\mathcal{P}}(y) \tag{1.4}$$

for $x < y$, where $u_n := a_n x + b_n$, $v_n := a_n y + b_n$; definitions of $D(u_n, v_n)$ and $D'(u_n)$ can be found in [5,7] and [10]. For the case that (1.3) holds with $\mathcal{P}$ a random variable, recently [5] showed that (1.4) still holds with $H(\mathcal{P}, x, y) = \mathbf{E}(G^{\mathcal{P}}(x) G^{1-\mathcal{P}}(y))$.

A closely related work to [10] is contribution [6], which considers a strongly dependent stationary Gaussian random sequence $\{X_n,\ n \geqslant 1\}$ with correlation $r_n = \mathbf{E}(X_1 X_{n+1})$ such that

$$\lim_{n\to\infty} r_n \ln n = \gamma \in [0, \infty). \tag{1.5}$$

In this case, $F = \Phi$, the distribution function of an $N(0,1)$ random variable, and therefore, (1.1) holds with norming constants a_n and b_n given by

$$a_n = \frac{1}{\sqrt{2\ln n}}, \qquad b_n = \sqrt{2\ln n} - \frac{\ln\ln n + \ln 4\pi}{2\sqrt{2\ln n}}. \tag{1.6}$$

Assuming that (1.5) holds, we have (see [6, 9, 14])

$$\lim_{n\to\infty} \mathbf{P}\bigl\{a_n^{-1}(M_n - b_n) \leqslant x\bigr\} = \mathbf{E}\bigl(\exp\bigl(-\exp(-x - \gamma + \sqrt{2\gamma} W)\bigr)\bigr), \tag{1.7}$$

where W is an $N(0,1)$ random variable. Clearly, when $\gamma = 0$, the limit distribution on the right-hand side of (1.7) is the Gumbel distribution $\Lambda(x) = \exp(-\exp(-x))$, $x \in \mathbb{R}$, which is shown in the seminal paper [1]. Commonly, the case $\gamma = 0$ is referred to as the case of weak dependence since the limit distribution of the maxima of the stationary process is the same as that of an iid sequence with underlying distribution function Φ.

We expect that for this case, again (1.4) holds under the general settings of [5], which is confirmed in the next section. If $\gamma > 0$, the limiting distribution of the maxima is a mixture distribution different from the Gumbel distribution, and thus, for that case, we cannot use the result of [5]. In order to overcome this difficulty, we shall borrow some ideas from [6], which cover the strong dependence case.

In the seminal paper [13], Piterbarg considered the joint approximation of the maximum of a stationary Gaussian process over a discrete and continuous grid of points. The results in [6] and [10] are motivated by the ideas and techniques developed in the aforementioned paper. Therefore, we shall refer in this contribution to the joint limit distribution of maxima of complete and incomplete samples such as (1.4) as the Piterbarg theorem.

In this paper, we are concerned only with stationary Gaussian sequences assuming that (1.5) holds with $\gamma \in [0, \infty]$. For the case of a weakly dependent stationary Gaussian sequence $\{X_n,\, n \geqslant 1\}$, i.e., condition (1.5) (or the so-called Berman condition) holds with $\gamma = 0$, then both conditions $D(u_n, v_n)$ and $D'(u_n)$ hold with the choice of constants a_n and b_n given by (1.6). Therefore, in view of [5], the Piterbarg theorem is valid (see Section 2).

Our main results show that the Piterbarg theorem also holds for the general case of $\gamma \in [0, \infty]$. Furthermore, we generalize the recent findings of [16], which are motivated by [6]. For some related work on asymptotic behavior of extremes of Gaussian sequences, see [2, 11].

Brief organization of the rest of the paper: Section 2 presents the main results; their proofs are relegated to Section 3.

2 Main results

In the sequel, let $\{X_n,\, n \geqslant 1\}$ be a standard stationary Gaussian sequence with underlying distribution function Φ and correlations $\{r_n,\, n \geqslant 1\}$ satisfying the dependence condition (1.5) with $\gamma \in [0, \infty]$. As in the Introduction, in order to derive the Piterbarg theorem, we shall assume further that the indicator random variables $\{\varepsilon_n,\, n \geqslant 1\}$ are independent of the Gaussian sequence and further (1.3) holds. Our first result is closely related to the result of [5].

Theorem 1. *Suppose that the stationary Gaussian sequence $\{X_n,\, n \geqslant 1\}$ is independent of indicator sequence $\{\varepsilon_n,\, n \geqslant 1\}$. If, further,* (1.3) *holds with some random variable $\mathcal{P}$, then, under the Berman condition,*

$$\lim_{n\to\infty} \mathbf{P}\big\{M_n(\varepsilon) \leqslant a_n x + b_n,\ M_n \leqslant a_n y + b_n\big\} = \mathbf{E}\big(\Lambda^{\mathcal{P}}(x)\Lambda^{1-\mathcal{P}}(y)\big)$$

for all real x, y with $x < y$ and constants a_n and b_n given by (1.6)*.*

Note, in passing, that the Berman condition implies the convergence of sample maxima $\{M_n,\, n \geqslant 1\}$ (after normalization) to a unit Gumbel random variable, whereas the above result implies that

$$\lim_{n\to\infty} \mathbf{P}\big\{M_n(\varepsilon) \leqslant a_n x + b_n\big\} = \mathbf{E}\big(\Lambda^{\mathcal{P}}(x)\big) \quad \forall x \in \mathbb{R},$$

and thus, we have the joint convergence in distribution

$$\left(\frac{M_n(\varepsilon) - b_n}{a_n}, \frac{M_n - b_n}{a_n}\right) \xrightarrow{\mathrm{d}} (\widetilde{\mathcal{M}}, \mathcal{M}), \quad n \to \infty, \tag{2.1}$$

where $(\widetilde{\mathcal{M}}, \mathcal{M})$ have joint distribution function $H(\mathcal{P}, x, y)$ defined by

$$H(\mathcal{P}, x, y) = \begin{cases} \mathbf{E}(\Lambda^{\mathcal{P}}(x)\Lambda^{1-\mathcal{P}}(y)) & \text{if } x < y, \\ \Lambda(y) & \text{otherwise.} \end{cases}$$

In [6] and later in [16], the limit of $\mathbf{P}\{M_n - M_n(\varepsilon) \leqslant a_n x,\ M_n(\varepsilon) - b_n \leqslant a_n y\}$ as $n \to \infty$ for any $x > 0$ and $y \in \mathbb{R}$ is derived. By the continuous mapping theorem the joint convergence in (2.1) thus implies the following corollary, which generalizes Theorem 1, Corollary 1, and Theorem 2 in [16].

Corollary 1. Under the assumptions of Theorem 1, we have the joint convergence in distribution

$$\left(\frac{M_n - M_n(\varepsilon)}{a_n}, \frac{M_n(\varepsilon) - b_n}{a_n}\right) \xrightarrow{\mathrm{d}} (\mathcal{M} - \widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}), \quad n \to \infty. \tag{2.2}$$

Next, we consider strongly dependent Gaussian sequences.

Theorem 2. *Suppose that the stationary Gaussian sequence* $\{X_n,\, n \geqslant 1\}$ *is independent of indicator sequence* $\{\varepsilon_n,\, n \geqslant 1\}$ *and* (1.3) *holds with some random variable* $\mathcal{P}$. *If, further,* (1.5) *holds with* $\gamma \in (0,\infty)$, *then, for all real* x, y *with* $x < y$, *we have*

$$\lim_{n\to\infty} \mathbf{P}\bigl\{M_n(\varepsilon) \leqslant a_n x + b_n,\ M_n \leqslant a_n y + b_n\bigr\} = H(\mathcal{P}, x, y),$$

where a_n *and* b_n *are given by* (1.6), *and*

$$H(\mathcal{P}, x, y) = \mathbb{E}\Biggl(\int_{-\infty}^{+\infty} \exp\bigl(-\mathcal{P}\exp(-x-\gamma+\sqrt{2\gamma}z) - (1-\mathcal{P})\exp(-y-\gamma+\sqrt{2\gamma}z)\bigr)\,\mathrm{d}\Phi(z)\Biggr). \quad (2.3)$$

Clearly, the above result can be stated as a joint convergence in distribution, i.e., we have again that (2.1) holds with $(\widetilde{\mathcal{M}}, \mathcal{M})$, which has joint distribution function $H(\mathcal{P}, x, y)$ given by (2.3) for all $x < y$, and $H(\mathcal{P}, x, y)$ equals the right-hand side of (1.7) for $x \geqslant y$. Consequently, we obtain the following result, which extends Theorem 3 in [16], where $\mathcal{P}$ is considered to be a constant.

Corollary 2. Under the assumptions of Theorem 2, the joint convergence in distribution in (2.2) holds, where $(\widetilde{\mathcal{M}}, \mathcal{M})$ has the joint distribution function $H(\mathcal{P}, x, y)$ given by (2.3).

Remark 1. In view of [6] (see also [12]), condition (1.5) can be slightly relaxed in the case of $\gamma \in (0,\infty)$. The result of Theorem 2 is still valid under the weaker condition stated in Eq. (5) in [6].

It is possible to have a joint convergence of sample maxima and that of incomplete sample maxima even if condition $D'(u_n)$ is not satisfied see [10]. As shown therein for a stationary non-Gaussian process related to the storage process, both maxima are completely dependent, which is a result expected in view of the findings of [4].

In Theorem 3 below, we investigate strongly dependent Gaussian sequences satisfying (1.5) with $\gamma = \infty$. We first recall a result known in the literature for the convergence of the sample maxima. Namely, under the conditions

(i) r_n is convex with $r_n = o(1)$ and (2.4)

(ii) $(r_n \ln n)^{-1}$ is monotone with $(r_n \ln n)^{-1} = o(1)$ (2.5)

as $n \to \infty$, [8] and [9] showed that, with $\tilde{b}_n := (1 - r_n)^{1/2} b_n$,

$$\lim_{n\to\infty} \mathbf{P}\{M_n - \tilde{b}_n \leqslant \sqrt{r_n}x\} = \Phi(x) \quad \forall x \in \mathbb{R}. \quad (2.6)$$

An extension of (2.6) to the Piterbarg max-discretization theorem is given in Corollary 2.2 of [15].

Our last result extends the above convergence as follows.

Theorem 3. *Suppose that the stationary Gaussian sequence* $\{X_n,\, n \geqslant 1\}$ *is independent of indicator sequence* $\{\varepsilon_n,\, n \geqslant 1\}$ *and* (1.3) *is valid with some random variable* $\mathcal{P} \in (0, 1]$. *If* (2.4) *and* (2.5) *hold, then, for all real* x, y, *we have*

$$\lim_{n\to\infty} \mathbf{P}\bigl\{M_n(\varepsilon) - \tilde{b}_n \leqslant \sqrt{r_n}x,\ M_n - \tilde{b}_n \leqslant \sqrt{r_n}y\bigr\} = \Phi\bigl(\min(x, y)\bigr),$$

where $\tilde{b}_n := (1 - r_n)^{1/2} b_n$ *with* b_n *given by* (1.6).

Again, we can cast the above result in the framework of joint convergence in distribution stated in (2.1), namely,

$$\left(\frac{M_n(\varepsilon)-\tilde{b}_n}{\sqrt{r_n}}, \frac{M_n-\tilde{b}_n}{\sqrt{r_n}}\right) \xrightarrow{\mathrm{d}} (W, W), \quad n \to \infty, \tag{2.7}$$

where W has the $N(0,1)$ distribution. Consequently, in the language of [6], we have

$$\left(\frac{M_n(\varepsilon)-\tilde{b}_n}{\sqrt{r_n}}, \frac{M_n-M_n(\varepsilon)}{\sqrt{r_n}}\right) \xrightarrow{\mathrm{d}} (W, 0), \quad n \to \infty. \tag{2.8}$$

3 Further results and proofs

In order to prove the main theorems, we need some auxiliary results. We borrow the following notation from [5]. Let $u_n(x) = a_n x + b_n$, $x \in \mathbb{R}$, and $\alpha = \{\alpha_n,\ n \geqslant 1\}$ be a nonrandom sequence taking values in $\{0,1\}$. For fixed k, let

$$K_s = \{j\colon\ (s-1)m + 1 \leqslant j \leqslant sm\}, \quad 1 \leqslant s \leqslant k,$$

where $m = [n/k]$. Further, for a random variable $\mathcal{P}$ such that $0 \leqslant \mathcal{P} \leqslant 1$ a.s., set

$$B_{t,k} = \left\{\omega\colon\ \mathcal{P}(\omega) \in \begin{cases} \left[0, \frac{1}{2^k}\right], & t = 0, \\ \left(\frac{t}{2^k}, \frac{t+1}{2^k}\right], & 0 < t \leqslant 2^k - 1 \end{cases}\right\},$$

and define

$$B_{t,k,\alpha,n} = \{\omega\colon\ \varepsilon_j(\omega) = \alpha_j,\ 1 \leqslant j \leqslant n\} \cap B_{t,k}.$$

In the following, C is a positive constant with values changing in different lines; we thus use C to omit the $O(1)$ notation.

Lemma 1. *Let $\{X_n^*,\ n \geqslant 1\}$ be a sequence of independent standard Gaussian random variables. Suppose that, for large n, the positive integers l are such that $k < l < m = [n/k]$ and $l = o(n)$. Then, for $x < y$,*

$$\left|\mathbf{P}\{M_n^*(\alpha) \leqslant u_n(x),\ M_n^* \leqslant u_n(y)\} - \prod_{s=1}^{k} \mathbf{P}\{M^*(K_s,\alpha) \leqslant u_n(x),\ M^*(K_s) \leqslant u_n(y)\}\right| \\ \leqslant (4k+2)l\bigl(1 - \Phi(u_n(x))\bigr)$$

uniformly for all $\alpha \in \{0,1\}^n$, where $M_n^* = \max\{X_j^*,\ 1 \leqslant j \leqslant n\}$, $M^*(K_s) = \max\{X_j^*,\ j \in K_s\}$,

$$M^*(K_s,\alpha) = \begin{cases} \max\{X_j^*,\ \alpha_j = 1,\ j \in K_s\} & \text{if } \sum_{j\in K_s} \alpha_j \geqslant 1, \\ -\infty & \text{otherwise}, \end{cases}$$

and

$$M_n^*(\alpha) = \begin{cases} \max\{X_j^*,\ \alpha_j = 1,\ 1 \leqslant j \leqslant n\} & \text{if } \sum_{j=1}^{n} \alpha_j \geqslant 1, \\ -\infty & \text{otherwise}. \end{cases}$$

Proof. First, we classify km integers into $2k$ consecutive intervals as follows. For large n, let l be integers such that $k < l < m$ and $l = o(n)$. Write

$$I_s = \{(s-1)m + 1, \ldots, sm - l\}, \qquad J_s = \{sm - l + 1, \ldots, sm\}$$

for $1 \leqslant s \leqslant k$, and set

$$I_{k+1} = \{(k-1)m + l + 1, \ldots, km\}, \qquad J_{k+1} = \{km + 1, \ldots, km + l\}.$$

Since $\{X_n^*,\ n \geqslant 1\}$ are independent, using the arguments similar to the proof of Lemma 4.3 in [10], we obtain the desired result. □

Lemma 2. *Under the conditions of Theorem* 2*, for $x < y$, we have*

$$\left|\mathbf{P}\{M_n(\alpha) \leqslant u_n(x),\ M_n \leqslant u_n(y)\} - \int_{-\infty}^{+\infty} \mathbf{P}\{M_n^*(\alpha) \leqslant v_n(x,z),\ M_n^* \leqslant v_n(y,z)\}\, \mathrm{d}\Phi(z)\right|$$
$$\leqslant Cn \sum_{k=1}^{n} |r_k - \rho_n| \exp\left(-\frac{u_n^2(x)}{1 + w_k}\right)$$

uniformly for all $\alpha \in \{0,1\}^n$, where $v_n(x,z) = (1-\rho_n)^{-1/2}(u_n(x) - \rho_n^{1/2} z)$, $\rho_n = \gamma/\ln n$, $w_k = \max\{|r_k|, \rho_n\}$, and C is some positive constant.

Proof. Let $\{\xi_{n,k},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ be a triangular array of standard Gaussian random variables with correlation $\rho_n = \gamma/\ln n$, and define

$$M_n^{\xi}(\alpha) = \begin{cases} \max\{\xi_{n,j},\ \alpha_j = 1,\ 1 \leqslant j \leqslant n\} & \text{if } \sum_{j=1}^{n} \alpha_j \geqslant 1, \\ -\infty & \text{otherwise,} \end{cases}$$

and $M_n^{\xi} = \max\{\xi_{n,k},\ 1 \leqslant k \leqslant n\}$. By Berman's inequality (see, e.g., [7] or [12]),

$$\left|\mathbf{P}\{M_n(\alpha) \leqslant u_n(x),\ M_n \leqslant u_n(y)\} - \mathbf{P}\{M_n^{\xi}(\alpha) \leqslant u_n(x),\ M_n^{\xi} \leqslant u_n(y)\}\right|$$
$$\leqslant Cn \sum_{k=1}^{n} |r_k - \rho_n| \exp\left(-\frac{u_n^2(x)}{1 + w_k}\right)$$

uniformly for all $\alpha \in \{0,1\}^n$, where $w_k = \max\{|r_k|, \rho_n\}$. According to the proof of Theorem 6.5.1 in [7],

$$\mathbf{P}\{M_n^{\xi}(\alpha) \leqslant u_n(x),\ M_n^{\xi} \leqslant u_n(y)\} = \int_{-\infty}^{+\infty} \mathbf{P}\{M_n^*(\alpha) \leqslant v_n(x,z),\ M_n^* \leqslant v_n(y,z)\}\, \mathrm{d}\Phi(z),$$

where $v_n(x,z) = (1-\rho_n)^{-1/2}(u_n(x) - \rho_n^{1/2} z)$, and thus, the claim follows. □

Lemma 3. *Let $\{Y_{n,k},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ be a triangular array of standard Gaussian sequences with correlation $\rho_k = (r_k - r_n)/(1 - r_n)$, where r_k satisfies* (2.4) *and* (2.5) *for $k = 1, 2, \ldots, n$. Suppose that $\{Y_{n,k},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ is independent of indicator sequence $\{\varepsilon_n,\ n \geqslant 1\}$. If, further,* (1.3) *holds with some random variable $\mathcal{P} \in (0,1]$, then, for all $\delta > 0$, we have*

$$\lim_{n\to\infty} \mathbf{P}\{M_n^Y(\varepsilon) \leqslant b_n - \delta r_n^{1/2}\} = 0,$$

where

$$M_n^Y(\varepsilon)=\begin{cases}\max\{Y_{n,j},\ \varepsilon_j=1,\ 1\leqslant j\leqslant n\} & \textit{if } \sum_{j=1}^n \varepsilon_j\geqslant 1,\\ -\infty & \textit{otherwise}.\end{cases}$$

Proof. Let $\{Z_{n,k},\ 1\leqslant k\leqslant n,\ n\geqslant 1\}$, $\{W_{n,k},\ 1\leqslant k\leqslant n,\ n\geqslant 1\}$ be two triangular arrays of standard Gaussian sequences with correlations defined by

$$\mathbf{E}(Z_{n,1}Z_{n,i+1})=\begin{cases}\rho_i, & 1\leqslant i\leqslant t(n),\\ \rho_{t(n)}, & i>t(n),\end{cases}\qquad \mathbf{E}(W_{n,1}W_{n,i+1})=\sigma_i=\begin{cases}\frac{\rho_i-\rho_{t(n)}}{1-\rho_{t(n)}}, & 1\leqslant i\leqslant t(n),\\ 0, & i>t(n),\end{cases}$$

respectively, where $t(n)=[n\exp(-(\ln n)^{1/2})]$. Suppose that the two Gaussian sequences are independent of the indicator sequence $\{\varepsilon_n,\ n\geqslant 1\}$. Define $M_n^Z(\varepsilon)$ and $M_n^W(\varepsilon)$ similarly to above, and let η be a standard Gaussian random variable independent of $\{W_{n,k},\ 1\leqslant k\leqslant n,\ n\geqslant 1\}$. Using Slepian's inequality (see, e.g., [12]), we have

$$\begin{aligned}&\mathbf{P}\{M_n^Y(\alpha)\leqslant b_n-\delta r_n^{1/2}\}\\ &\leqslant \mathbf{P}\{M_n^Z(\alpha)\leqslant b_n-\delta r_n^{1/2}\}=\mathbf{P}\{(1-\rho_{t(n)})^{1/2}M_n^W(\alpha)+\rho_{t(n)}^{1/2}\eta\leqslant b_n-\delta r_n^{1/2}\}\\ &=\int_{-\infty}^{+\infty}\mathbf{P}\{M_n^W(\alpha)\leqslant \big(b_n-\delta r_n^{1/2}-\rho_{t(n)}^{1/2}z\big)(1-\rho_{t(n)})^{-1/2}\}\,\mathrm{d}\Phi(z)\\ &\leqslant \Phi\bigg(-\frac{\delta r_n^{1/2}}{2\rho_{t(n)}^{1/2}}\bigg)+\mathbf{P}\bigg\{M_n^W(\alpha)\leqslant\bigg(b_n-\frac{\delta r_n^{1/2}}{2}\bigg)(1-\rho_{t(n)})^{-1/2}\bigg\}.\end{aligned}$$

Further, using Berman's inequality, we obtain

$$\begin{aligned}&\bigg|\mathbf{P}\bigg\{M_n^W(\alpha)\leqslant\bigg(b_n-\frac{\delta r_n^{1/2}}{2}\bigg)(1-\rho_{t(n)})^{-1/2}\bigg\}-\mathbf{P}\bigg\{M_n^*(\alpha)\leqslant\bigg(b_n-\frac{\delta r_n^{1/2}}{2}\bigg)(1-\rho_{t(n)})^{-1/2}\bigg\}\bigg|\\ &\leqslant Cn\sum_{i=1}^{t(n)}\sigma_i\exp\bigg(-\frac{(b_n-\delta r_n^{1/2}/2)^2}{(1+\sigma_i)(1-\rho_{t(n)})}\bigg)=: c_n.\end{aligned}$$

Hence, by the total probability formula,

$$\begin{aligned}\mathbf{P}\{M_n^Y(\varepsilon)\leqslant b_n-\delta r_n^{1/2}\}&=\sum_{t=0}^{2^k-1}\sum_{\alpha\in\{0,1\}^n}\mathbf{P}\{M_n^Y(\alpha)\leqslant b_n-\delta r_n^{1/2}\}\mathbf{P}(B_{t,k,\alpha,n})\\ &\leqslant \Phi\bigg(-\frac{\delta r_n^{1/2}}{2\rho_{t(n)}^{1/2}}\bigg)+\mathbf{P}\bigg\{M_n^*(\varepsilon)\leqslant\bigg(b_n-\frac{\delta r_n^{1/2}}{2}\bigg)(1-\rho_{t(n)})^{-1/2}\bigg\}+c_n.\end{aligned}$$

By properties (2.4) and (2.5) of $\{r_n,\ n\geqslant 1\}$, the useful facts (see [9, p. 9])

$$\lim_{n\to\infty}\frac{r_n}{\rho_{t(n)}}=\infty,\qquad \lim_{n\to\infty}\frac{b_n\rho_{t(n)}}{r_n^{1/2}}=0$$

imply that

$$\lim_{n\to\infty} \Phi\biggl(-\frac{\delta r_n^{1/2}}{2\rho_{t(n)}^{1/2}}\biggr) = 0$$

and

$$\mathbf{P}\biggl\{M_n^*(\varepsilon) \leqslant \biggl(b_n - \frac{\delta r_n^{1/2}}{2}\biggr)(1-\rho_{t(n)})^{-1/2}\biggr\} \leqslant \mathbf{P}\bigl\{M_n^*(\varepsilon) \leqslant -a_n A + b_n\bigr\}$$

for arbitrary positive number A and large n. By Corollary 1 in [5],

$$\limsup_{n\to\infty} \mathbf{P}\biggl\{M_n^*(\varepsilon) \leqslant \biggl(b_n - \frac{\delta r_n^{1/2}}{2}\biggr)(1-\rho_{t(n)})^{-1/2}\biggr\} \leqslant \mathbf{E}\bigl(\Lambda^{\mathcal{P}}(-A)\bigr).$$

Letting $A \to \infty$, we have

$$\lim_{n\to\infty} \mathbf{P}\biggl\{M_n^*(\varepsilon) \leqslant \biggl(b_n - \frac{\delta r_n^{1/2}}{2}\biggr)(1-\rho_{t(n)})^{-1/2}\biggr\} = 0.$$

By the arguments of [8, pp. 187–188] we have that $\lim_{n\to\infty} c_n = 0$, and hence the claim follows. $\square$

Proof of Theorem 1. For a stationary Gaussian sequence, the conditions $D(u_n, v_n)$ and $D'(u_n)$ hold when the correlations satisfy (1.5) with $\gamma = 0$; see Lemma 4.4.1 in [7] for details. Hence, according to Theorem 1.1 in [5], we obtain the desired result. $\square$

Proof of Theorem 2. Let $\Psi(n,x,z) = n(1-\Phi(v_n(x,z))$ with $v_n(x,z) = (1-\rho_n)^{-1/2}(u_n(x) - \rho_n^{1/2} z)$. Note that

$$\begin{aligned}
&\Biggl|\mathbf{P}\bigl\{M_n(\varepsilon) \leqslant u_n(x),\ M_n \leqslant u_n(y)\bigr\} - \mathbb{E}\Biggl(\int_{-\infty}^{+\infty} \prod_{s=1}^{k}\biggl(1 - \frac{\mathcal{P}\Psi(n,x,z) + (1-\mathcal{P})\Psi(n,y,z)}{k}\biggr)\,\mathrm{d}\Phi(z)\Biggr)\Biggr| \\
&\quad\leqslant \sum_{t=0}^{2^k-1} \sum_{\alpha\in\{0,1\}^n} \mathbb{E}\Biggl(\Biggl|\mathbf{P}\bigl\{M_n(\alpha) \leqslant u_n(x),\ M_n \leqslant u_n(y)\bigr\} \\
&\qquad - \int_{-\infty}^{+\infty} \prod_{s=1}^{k}\biggl(1 - \frac{\mathcal{P}\Psi(n,x,z) + (1-\mathcal{P})\Psi(n,y,z)}{k}\biggr)\,\mathrm{d}\Phi(z)\Biggr|\mathbf{I}(B_{t,k,\alpha,n})\Biggr) \\
&\quad\leqslant E_1 + E_2 + E_3 + E_4,
\end{aligned}$$

where

$$\begin{aligned}
E_1 = \sum_{t=0}^{2^k-1} \sum_{\alpha\in\{0,1\}^n} \mathbf{E}\Biggl(\Biggl|&\mathbf{P}\bigl\{M_n(\alpha) \leqslant u_n(x),\ M_n \leqslant u_n(y)\bigr\} \\
&- \int_{-\infty}^{+\infty} \mathbf{P}\bigl\{M_n^*(\alpha) \leqslant v_n(x,z),\ M_n^* \leqslant v_n(y,z)\bigr\}\,\mathrm{d}\Phi(z)\Biggr|\mathbf{I}(B_{t,k,\alpha,n})\Biggr),
\end{aligned}$$

$$E_2 = \sum_{t=0}^{2^k-1} \sum_{\alpha \in \{0,1\}^n} \mathbf{E}\Bigg(\int_{-\infty}^{+\infty} \bigg| \mathbf{P}\big\{ M_n^*(\alpha) \leqslant v_n(x,z),\ M_n^* \leqslant v_n(y,z) \big\} - \prod_{s=1}^{k} \mathbf{P}\big\{ M^*(K_s,\alpha) \leqslant v_n(x,z),\ M^*(K_s) \leqslant v_n(y,z) \big\} \bigg| \mathrm{d}\Phi(z)\, \mathbf{I}(B_{t,k,\alpha,n}) \Bigg),$$

$$E_3 = \sum_{t=0}^{2^k-1} \sum_{\alpha \in \{0,1\}^n} \mathbf{E}\Bigg(\int_{-\infty}^{+\infty} \bigg| \prod_{s=1}^{k} \mathbf{P}\big\{ M^*(K_s,\alpha) \leqslant v_n(x,z),\ M^*(K_s) \leqslant v_n(y,z) \big\} - \prod_{s=1}^{k} \bigg(1 - \frac{t/2^k \Psi(n,x,z) + (1-t/2^k)\Psi(n,y,z)}{k} \bigg) \bigg| \mathrm{d}\Phi(z)\, \mathbf{I}(B_{t,k,\alpha,n}) \Bigg),$$

and

$$E_4 = \sum_{t=0}^{2^k-1} \sum_{\alpha \in \{0,1\}^n} \mathbf{E}\Bigg(\int_{-\infty}^{+\infty} \bigg| \prod_{s=1}^{k} \bigg(1 - \frac{\mathcal{P}\Psi(n,x,z) + (1-\mathcal{P})\Psi(n,y,z)}{k} \bigg) - \prod_{s=1}^{k} \bigg(1 - \frac{t/2^k \Psi(n,x,z) + (1-t/2^k)\Psi(n,y,z)}{k} \bigg) \bigg| \mathrm{d}\Phi(z)\, \mathbf{I}(B_{t,k,\alpha,n}) \Bigg).$$

Using Lemma 2 and Lemma 6.4.1 in [7], we have

$$E_1 \leqslant Cn \sum_{i=1}^{n} |r_i - \rho_n| \exp\bigg(-\frac{u_n^2(x)}{1+w_i} \bigg) \to 0 \tag{3.1}$$

as $n \to \infty$. For E_2, according to Lemma 1, we have

$$E_2 \leqslant (4k+2)\frac{l}{n} \int_{-\infty}^{+\infty} \Psi(n,x,z)\, \mathrm{d}\Phi(z).$$

According to the proof of Theorem 6.5.1 in [7], we have $v_n(x,z) = u_n(x + \gamma - \sqrt{2\gamma}z) + o(a_n)$, and thus,

$$\lim_{n\to\infty} \Psi(n,x,z) = \exp(-x - \gamma + \sqrt{2\gamma}z) =: h(x,z,\gamma).$$

Combined with $l = o(n)$ as $n \to \infty$, the dominated convergence theorem yields

$$\lim_{n\to\infty} E_2 = 0. \tag{3.2}$$

Next, using Lemma 3 in [5], we have

$$E_3 \leqslant \sum_{t=0}^{2^k-1} \sum_{\alpha \in \{0,1\}^n} \mathbf{E}\Bigg(\int_{-\infty}^{+\infty} \sum_{s=1}^{k} \bigg| \mathbf{P}\big\{ M^*(K_s,\alpha) \leqslant v_n(x,z),\ M^*(K_s) \leqslant v_n(y,z) \big\}$$

$$
-\left(1-\frac{t/2^k\Psi(n,x,z)+(1-t/2^k)\Psi(n,y,z)}{k}\right)\bigg|\,\mathrm{d}\Phi(z)\,\mathbf{I}(B_{t,k,\alpha,n})\Bigg)
$$

$$
\leqslant\sum_{t=0}^{2^k-1}\sum_{\alpha\in\{0,1\}^n}\mathbf{E}\Bigg(\int_{-\infty}^{+\infty}\sum_{s=1}^{k}\bigg|\frac{\sum_{j\in K_s}\alpha_j}{m}-\frac{t}{2^k}\bigg|\frac{n(\Phi(v_n(y,z))-\Phi(v_n(x,z)))}{k}\,\mathrm{d}\Phi(z)\,\mathbf{I}(B_{t,k,\alpha,n})\Bigg)
$$

$$
+\frac{1}{k}\int_{-\infty}^{+\infty}\big(\Psi(n,x,z)\big)^2\,\mathrm{d}\Phi(z)
$$

$$
=\sum_{t=0}^{2^k-1}\sum_{s=1}^{k}\mathbf{E}\bigg(\bigg|\sum_{j\in K_s}\frac{\varepsilon_j}{m}-\frac{t}{2^k}\bigg|\mathbf{I}(B_{t,k})\bigg)\int_{-\infty}^{+\infty}\frac{n(\Phi(v_n(y,z))-\Phi(v_n(x,z)))}{k}\,\mathrm{d}\Phi(z)
$$

$$
+\frac{1}{k}\int_{-\infty}^{+\infty}\big(\Psi(n,x,z)\big)^2\,\mathrm{d}\Phi(z)
$$

$$
\leqslant\sum_{s=1}^{k}\bigg(\mathbf{E}\bigg|\sum_{j\in K_s}\frac{\varepsilon_j}{m}-\mathcal{P}\bigg|+\frac{1}{2^k}\bigg)\int_{-\infty}^{+\infty}\frac{n(\Phi(v_n(y,z))-\Phi(v_n(x,z)))}{k}\,\mathrm{d}\Phi(z)
$$

$$
+\frac{1}{k}\int_{-\infty}^{+\infty}\big(\Psi(n,x,z)\big)^2\,\mathrm{d}\Phi(z)
$$

$$
\leqslant\sum_{s=1}^{k}\bigg[2(2s-1)\bigg(d\bigg(\frac{S_{sm}}{sm},\mathcal{P}\bigg)+d\bigg(\frac{S_{(s-1)m}}{(s-1)m},\mathcal{P}\bigg)\bigg)+\frac{1}{2^k}\bigg]\int_{-\infty}^{+\infty}\frac{\Psi(n,x,z)-\Psi(n,y,z)}{k}\,\mathrm{d}\Phi(z)
$$

$$
+\frac{1}{k}\int_{-\infty}^{+\infty}\big(\Psi(n,x,z)\big)^2\,\mathrm{d}\Phi(z),
$$

where $d(X,Y)$ stands for the Ky Fan metric, i.e., $d(X,Y)=\inf\{\epsilon\colon\ \mathbf{P}\{|X-Y|>\epsilon\}<\epsilon\}$. Since

$$
\lim_{m\to\infty}d\bigg(\frac{S_{sm}}{sm},\mathcal{P}\bigg)=0,
$$

taking the limit as $n\to\infty$ and then as $m\to\infty$, we obtain

$$
\limsup_{n\to\infty}E_3\leqslant\frac{1}{2^k}\int_{-\infty}^{+\infty}\big(h(x,z,\gamma)-h(y,z,\gamma)\big)\,\mathrm{d}\Phi(z)+\frac{1}{k}\int_{-\infty}^{+\infty}h^2(x,z,\gamma)\,\mathrm{d}\Phi(z). \tag{3.3}
$$

For E_4, we have

$$
E_4\leqslant\sum_{t=0}^{2^k-1}\sum_{\alpha\in\{0,1\}^n}\mathbf{E}\Bigg(\int_{-\infty}^{+\infty}\sum_{s=1}^{k}\bigg|\mathcal{P}-\frac{t}{2^k}\bigg|\frac{\Psi(n,y,z)+\Psi(n,x,z)}{k}\,\mathrm{d}\Phi(z)\,\mathbf{I}(B_{t,k,\alpha,n})\Bigg)
$$

$$= \int_{-\infty}^{+\infty} \big(\Psi(n,y,z) + \Psi(n,x,z)\big)\,\mathrm{d}\Phi(z) \sum_{t=0}^{2^k-1} \mathbf{E}\bigg(\bigg|\mathcal{P} - \frac{t}{2^k}\bigg|\mathbf{I}(B_{t,k})\bigg)$$

$$\leqslant \int_{-\infty}^{+\infty} \frac{\Psi(n,y,z) + \Psi(n,x,z)}{2^k}\,\mathrm{d}\Phi(z)$$

$$\to \frac{1}{2^k} \int_{-\infty}^{+\infty} \big(h(x,z,\gamma) + h(y,z,\gamma)\big)\,\mathrm{d}\Phi(z) \tag{3.4}$$

as $n \to \infty$. Hence, combining (3.1)–(3.4), we have

$$\limsup_{n\to\infty} \Bigg| \mathbf{P}\big\{M_n(\varepsilon) \leqslant u_n(x),\ M_n \leqslant u_n(y)\big\} - \mathbf{E}\Bigg(\int_{-\infty}^{+\infty} \bigg(1 - \frac{\mathcal{P}\exp(-x-\gamma+\sqrt{2\gamma}z) + (1-\mathcal{P})\exp(-y-\gamma+\sqrt{2\gamma}z)}{k}\bigg)^k \mathrm{d}\Phi(z)\Bigg)\Bigg|$$

$$\leqslant \frac{1}{2^{k-1}} \int_{-\infty}^{+\infty} h(x,z,\gamma)\,\mathrm{d}\Phi(z) + \frac{1}{k}\int_{-\infty}^{+\infty} h^2(x,z,\gamma)\,\mathrm{d}\Phi(z).$$

The claimed result follows by letting $k \to \infty$. $\square$

Proof of Theorem 3. We next show that

$$\lim_{n\to\infty} \mathbf{P}\big\{r_n^{-1/2}\big(M_n(\varepsilon) - (1-r_n)^{1/2}b_n\big) \leqslant x\big\} = \Phi(x) \quad \forall x \in \mathbb{R}. \tag{3.5}$$

Let events $B_{t,k,\alpha,n}$ be defined as before. Since, by assumption, $\mathcal{P} > 0$ and the indicator random sequence $\{\varepsilon_n,\ n \geqslant 1\}$ is independent of $\{X_n,\ n \geqslant 1\}$ for any $x \in \mathbb{R}$, we have

$$\mathbf{P}\big\{r_n^{-1/2}\big(M_n(\varepsilon) - (1-r_n)^{12}b_n\big) \leqslant x\big\} = \sum_{t=0}^{2^k-1} \sum_{\alpha\in\{0,1\}^n} P(n,\alpha)\mathbf{P}(B_{t,k,\alpha,n}),$$

where $P(n,\alpha) = \mathbf{P}\{r_n^{-1/2}(M_n(\alpha) - (1-r_n)^{1/2}b_n) \leqslant x\}$. Applying Slepian's inequality, we further have

$$P(n,\alpha) = \int_{-\infty}^{+\infty} \mathbf{P}\big\{M_n^Y(\alpha) \leqslant b_n + r_n^{12}(1-r_n)^{-12}(x-z)\big\}\,\mathrm{d}\Phi(z)$$

$$\geqslant \int_{-\infty}^{+\infty} \mathbf{P}\big\{M_n^*(\alpha) \leqslant b_n + r_n^{1/2}(1-r_n)^{-1/2}(x-z)\big\}\,\mathrm{d}\Phi(z)$$

$$\geqslant \mathbf{P}\big\{M_n^*(\alpha) \leqslant b_n + r_n^{1/2}(1-r_n)^{-1/2}\delta\big\}\Phi(x-\delta)$$

for any $\delta > 0$. Since $\lim_{n\to\infty} a_n^{-1} r_n^{1/2} = \infty$, there exists sufficiently large A such that, for all n large,

$$\begin{aligned}
&\sum_{t=0}^{2^k-1} \sum_{\alpha\in\{0,1\}^n} P(n,\alpha)\mathbf{P}(B_{t,k,\alpha,n}) \\
&\quad \geqslant \Phi(x-\delta) \sum_{t=0}^{2^k-1} \sum_{\alpha\in\{0,1\}^n} \mathbf{P}\bigl\{M_n^*(\alpha) \leqslant b_n + r_n^{1/2}(1-r_n)^{-1/2}\delta\bigr\}\mathbf{P}(B_{t,k,\alpha,n}) \\
&\quad = \Phi(x-\delta)\mathbf{P}\bigl\{M_n^*(\varepsilon) \leqslant b_n + r_n^{1/2}(1-r_n)^{-1/2}\delta\bigr\} \geqslant \Phi(x-\delta)\mathbf{P}\bigl\{M_n^*(\varepsilon) \leqslant b_n + a_n A\delta\bigr\} \\
&\quad \geqslant \Phi(x-\delta)\mathbf{P}\bigl\{M_n^* \leqslant b_n + a_n A\delta\bigr\}.
\end{aligned}$$

Clearly, since

$$\lim_{A\to\infty} \lim_{n\to\infty} \mathbf{P}\bigl\{M_n^* \leqslant b_n + a_n A\delta\bigr\} = \lim_{A\to\infty} \exp\bigl(-\exp(-A\delta)\bigr) = 1,$$

we have

$$\liminf_{n\to\infty} \mathbf{P}\bigl\{r_n^{-1/2}\bigl(M_n(\varepsilon) - (1-r_n)^{1/2}b_n\bigr) \leqslant x\bigr\} \geqslant \Phi(x-\delta).$$

Next, we derive the upper bound. Note that

$$\begin{aligned}
P(n,\alpha) &\leqslant \int_{-\infty}^{+\infty} \mathbf{P}\bigl\{M_n^Y(\alpha) \leqslant b_n + r_n^{1/2}(1-r_n)^{-1/2}(x-z)\bigr\}\,\mathrm{d}\Phi(z) \\
&\leqslant \Phi(x+\delta) + \mathbf{P}\bigl\{M_n^Y(\alpha) \leqslant b_n - r_n^{1/2}(1-r_n)^{-1/2}\delta\bigr\},
\end{aligned}$$

implying

$$\mathbf{P}\bigl\{r_n^{-1/2}\bigl(M_n(\varepsilon) - (1-r_n)^{1/2}b_n\bigr) \leqslant x\bigr\} \leqslant \Phi(x+\delta) + \mathbf{P}\bigl\{M_n^Y(\varepsilon) \leqslant b_n - r_n^{1/2}(1-r_n)^{-1/2}\delta\bigr\}.$$

Using Lemma 3, we obtain

$$\limsup_{n\to\infty} \mathbf{P}\bigl\{r_n^{-1/2}\bigl(M_n(\varepsilon) - (1-r_n)^{1/2}b_n\bigr) \leqslant x\bigr\} \leqslant \Phi(x+\delta),$$

and hence, (3.5) follows by letting $\delta \downarrow 0$. Next note that, for any x, y,

$$\begin{aligned}
p_n(x,y) &:= \mathbf{P}\bigl\{r_n^{-1/2}\bigl(M_n(\varepsilon) - (1-r_n)^{1/2}b_n\bigr) \leqslant x,\ r_n^{-1/2}\bigl(M_n - (1-r_n)^{1/2}b_n\bigr) \leqslant y\bigr\} \\
&\leqslant \mathbf{P}\bigl\{r_n^{-1/2}\bigl(M_n(\varepsilon) - (1-r_n)^{1/2}b_n\bigr) \leqslant x\bigr\}
\end{aligned}$$

and, further, for $x < y$,

$$\begin{aligned}
&\mathbf{P}\bigl\{r_n^{-1/2}\bigl(M_n(\varepsilon) - (1-r_n)^{1/2}b_n\bigr) \leqslant x\bigr\} \\
&\quad \leqslant p_n(x,y) + \mathbf{P}\bigl\{r_n^{-1/2}\bigl(M_n(\varepsilon) - (1-r_n)^{1/2}b_n\bigr) \leqslant y\bigr\} - \mathbf{P}\bigl\{r_n^{-1/2}\bigl(M_n - (1-r_n)^{1/2}b_n\bigr) \leqslant y\bigr\} \\
&\quad = p_n(x,y) + o(1),
\end{aligned}$$

where the last claim above follows directly from the fact that (see (2.6))

$$\lim_{n\to\infty} \mathbf{P}\bigl\{r_n^{-1/2}\bigl(M_n - (1-r_n)^{1/2}b_n\bigr) \leqslant x\bigr\} = \Phi(x) \quad \forall x \in \mathbb{R}.$$

Consequently, for $x < y$, we have $p_n(x,y) = \mathbf{P}\{r_n^{-1/2}(M_n(\varepsilon) - (1-r_n)^{1/2}b_n) \leqslant x\} + o(1)$, and thus, the proof is complete. $\square$

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